

# Inhomogeneous High Frequency Expansion–Free Gravitational Waves

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## Abstract

We describe a natural inhomogeneous generalization of high frequency plane gravitational waves. The waves are high frequency waves of the Kundt type whose null propagation direction in space–time has vanishing expansion, twist and shear but is not covariantly constant. The introduction of a cosmological constant is discussed in some detail and a comparison is made with high frequency gravity waves having wave fronts homeomorphic to 2–spheres.

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# 1 Introduction

Two important special families of gravitational waves, from the point of view of modelling astrophysical processes, are bursts of gravitational radiation (which may be accompanied by matter travelling with the speed of light) and high frequency gravitational waves. For an overview of the former see [1]. This paper is concerned with establishing the line-elements of the space-time models of the gravitational fields of high frequency gravitational waves. For simplicity we consider monochromatic waves here. These are the basic building blocks for the study of the interaction of such waves in General Relativity and for constructing model systems responsible for such waves. They could be used to further develop the interesting ideas on the self-interaction of gravitational waves in [2] and [3] for example.

Einstein's theory of General Relativity is similar to Maxwell's theory of Electromagnetism in permitting plane wave solutions which are both homogeneous and inhomogeneous. Einstein's theory does not however permit exactly spherical waves (on account of the Birkhoff theorem) but it does of course permit waves from isolated sources having wave fronts which are homeomorphic to 2-spheres. A physically important subfield of the theory of waves in General Relativity is concerned with modelling the gravitational fields due to high frequency or short wavelength gravitational waves. The generation of such waves by a high frequency compact source in the field of a black hole, for example, and the influence of the background curvature on the wave propagation is studied in [4] (for an alternative mechanism generating such waves see [5]–[9]; for a complementary approach see Ellis [10]). Homogeneous, plane, high frequency gravitational waves are well known [11] and so this paper is primarily concerned with providing examples of high frequency, *inhomogeneous*, non-expanding gravitational waves and comparing these with the corresponding waves having roughly spherical wave fronts. The paper is organised as follows: In section 2 high frequency, homogeneous plane waves are described as a means of introducing our point of view. The generalization to *inhomogeneous* waves is given in section 3. If the inhomogeneous waves of section 3 are *pp*-waves (plane fronted waves with parallel rays) then the description of their high frequency version in the presence of a cosmological constant is given in section 4. The resulting high frequency waves are not *pp*-waves since their propagation direction in space-time is no longer covariantly constant in the presence of a cosmological constant. This is followed in section 5 by a comparison with high frequency 'spherical' waves. To make the paper as self contained as possible we provide calculational details in appendices.

## 2 Homogeneous Waves

The line-element of the space-time model of the (approximately) vacuum gravitational field due to a train of short wavelength or high frequency homogeneous, plane gravitational waves can be written in the form [11]

$$ds^2 = -2 P_\lambda(u)^{-2} |d\zeta + \lambda \bar{W}(u) \sin \frac{u}{\lambda} d\bar{\zeta}|^2 + 2 du dr . \quad (2.1)$$

Here  $\zeta$  is a complex coordinate with complex conjugate here and throughout indicated by a bar. The coordinates  $r, u$  are real.  $P_\lambda$  is a real-valued function of the coordinate  $u$  with a dependence on the real parameter  $\lambda \geq 0$  on account of Einstein's vacuum field equations.  $W$  is an arbitrary complex-valued function of the real coordinate  $u$ . The hypersurfaces  $u = \text{constant}$  are null and are generated by the null geodesic integral curves of the vector field  $k^a \partial/\partial x^a = \partial/\partial r$  with  $r$  and affine parameter along them. This vector is in fact covariantly constant and the null hypersurfaces are the histories of the plane wave fronts of the gravitational waves. The parameter  $\lambda$  will play the role of the wavelength of the gravitational waves and we shall assume that  $\lambda$  is small. For small  $\lambda$  the vacuum field equations are approximately satisfied by the metric tensor given via the line-element (2.1) in the sense that the Ricci tensor satisfies

$$R_{ab} = O(\lambda) , \quad (2.2)$$

provided  $P_\lambda(u)$  satisfies

$$- \dot{H}_\lambda + H_\lambda^2 + |W|^2 \sin^2 \frac{u}{\lambda} = 0 , \quad (2.3)$$

where the dot indicates differentiation with respect to  $u$  and

$$H_\lambda = P_\lambda^{-1} \dot{P}_\lambda . \quad (2.4)$$

The components of the Riemann curvature tensor in Newman-Penrose notation are denoted by  $\Psi_A$  with  $A = 0, 1, 2, 3, 4$  and when calculated with the metric tensor given via (2.1) we find that  $\Psi_A = O(\lambda)$  for  $A = 0, 1, 2, 3$  and

$$\Psi_4 = \lambda^{-1} W \sin \frac{u}{\lambda} + O(\lambda^0) . \quad (2.5)$$

Thus for small  $\lambda$  the Riemann tensor is type N in the Petrov classification having  $\partial/\partial r$  as degenerate principal null direction. Equation (2.5) indicates a profile for these waves which has large amplitude and short wavelength. Using the Riemann-Lebesgue theorem [11] (described in Appendix A) one can deduce from (2.3) that for small  $\lambda$

$$P_\lambda = P_0 + \frac{1}{8} \lambda^2 P_0 |W|^2 \cos \frac{2u}{\lambda} + O(\lambda^3) , \quad (2.6)$$

with  $P_0 = \lim_{\lambda \rightarrow 0} P_\lambda$  satisfying the differential equation

$$-\dot{H}_0 + H_0^2 + \frac{1}{2}|W|^2 = 0 , \quad (2.7)$$

with  $H_0 = P_0^{-1}\dot{P}_0$ . We observe that there is no  $O(\lambda)$ -term in (2.6) but the  $O(\lambda^2)$ -term is necessary in order to satisfy (2.2) and thus (2.3). Now the line-element (2.1) naturally splits, for small  $\lambda$ , into the form

$$ds^2 = d\hat{s}^2 + O(\lambda) , \quad (2.8)$$

with

$$d\hat{s}^2 = -2 P_0^{-2} |d\zeta|^2 + 2 du dr . \quad (2.9)$$

This ‘background’ line-element has Ricci tensor

$$\hat{R}_{ab} = -|W|^2 k_a k_b , \quad (2.10)$$

on account of (2.7). We note that  $k^a$  given following (2.1) above is equivalently given by the 1-form  $k_a dx^a = du$ . The lambda dependence in the equations (2.2) and (2.5) and the algebraic form of the background Ricci tensor (2.10) are what one expects in general for high frequency gravitational waves following the pioneering work of Isaacson [12][13] (see also [14], [15]). The coordinates  $\zeta, \bar{\zeta}$  are intrinsic to the histories of the plane wave fronts and the absence of the dependence of any functions on these coordinates in the example given here reflects the fact that the waves here are *homogeneous*.

### 3 Inhomogeneous Generalization

The generalization to *inhomogeneous*, high frequency non-expanding waves which we wish to consider here is expressed by the line-element:

$$ds^2 = -2 P_\lambda^{-2} |d\zeta + \lambda P_\lambda^2 \bar{W}(\bar{\zeta}, u) \sin \frac{u}{\lambda} d\bar{\zeta}|^2 + 2 du dr + (S_\lambda - 2 r H_\lambda) du^2 . \quad (3.1)$$

Here  $P_\lambda(\zeta, \bar{\zeta}, u)$  is a real-valued function satisfying

$$\Delta_\lambda \log P_\lambda = 0 , \quad \Delta_\lambda = 2 P_\lambda^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} , \quad (3.2)$$

and

$$H_\lambda = P_\lambda^{-1} \dot{P}_\lambda . \quad (3.3)$$

Also  $W(\zeta, u)$  is an arbitrary analytic function of its arguments while  $S_\lambda(\zeta, \bar{\zeta}, u)$  is a real-valued function satisfying

$$-\dot{H}_\lambda + 2H_\lambda^2 - \frac{1}{4}\Delta_\lambda S_\lambda + P_\lambda^4 |W|^2 \sin^2 \frac{u}{\lambda} = 0 . \quad (3.4)$$

Equation (3.4) is the analogue here of (2.3) and ensures that the metric given via the line-element (3.1) satisfies (2.2). Using the Riemann–Lebesgue theorem [11] (see Appendix A) one deduces from (3.4) that for small  $\lambda$

$$P_\lambda = P_0 + \frac{1}{8}\lambda^2 P_0^5 |W|^2 \cos \frac{2u}{\lambda} + O(\lambda^3) , \quad (3.5)$$

with  $P_0$  satisfying the differential equation

$$-\dot{H}_0 + 2H_0^2 - \frac{1}{4}\Delta_0 S_0 + \frac{1}{2}P_0^4 |W|^2 = 0 , \quad (3.6)$$

and the subscript zero denoting that the limit  $\lambda \rightarrow 0$  has been taken. The Newman–Penrose components of the Riemann curvature tensor calculated with the metric tensor given via (3.1) now satisfy  $\Psi_A = O(\lambda)$  for  $A = 0, 1, 2$  and

$$\Psi_3 = -P_0 \frac{\partial H_0}{\partial \zeta} + O(\lambda) , \quad (3.7)$$

$$\Psi_4 = \lambda^{-1} P_0^2 W \sin \frac{u}{\lambda} + O(\lambda^0) . \quad (3.8)$$

Thus for small  $\lambda$  the space–time with line–element (3.1) is a vacuum space–time (on account of (2.2)) and is type N in the Petrov classification (since for small  $\lambda$  the Riemann curvature tensor is dominated by the Newman–Penrose component  $\Psi_4$ ) and thus is a model of the gravitational field of low wavelength, high amplitude gravitational waves. These waves are inhomogeneous plane waves in the sense of Kundt [16] because their null propagation direction in space–time  $k^a \partial / \partial x^a = \partial / \partial r$  is twist–free, shear–free and expansion–free. They are in fact a subclass of such space–times because the covariant null vector field  $k_a dx^a = du$  in the space–time with line–element (3.1) has covariant derivative, with respect to the Riemannian connection calculated with the metric tensor given via (3.1) (and indicated by a stroke), of the form

$$k_{a|b} = H_\lambda k_a k_b . \quad (3.9)$$

Vacuum space–times admitting a null vector field satisfying an equation of this form were first discussed in detail in [17] and [18]. We see from (3.2) that  $P_\lambda = |h_\lambda(\zeta, u)|$  for some analytic function  $h_\lambda(\zeta, u)$ . The special case

$P_0 = 1$  can be realized with  $h_\lambda(\zeta, u) = 1 + O(\lambda^2)$  on account of (3.5). This leads to  $H_\lambda = O(\lambda)$  and  $\dot{H}_\lambda = O(\lambda^0)$  and so (3.9) reads  $k_{a|b} = O(\lambda)$ . Hence for small  $\lambda$  the inhomogeneous high frequency waves in this case are the so-called *pp*-waves [19] (plane-fronted waves with parallel rays). We will consider this special case in section 4 in order to describe the introduction of a cosmological constant. The introduction of a cosmological constant in the general case in which (3.9) holds is an interesting nontrivial topic for further study.

The line-element (3.1) can be written, for small  $\lambda$ , as a background plus a small perturbation as in (2.8) but now with

$$ds^2 = -2 P_0^{-2} |d\zeta|^2 + 2 du dr + (S_0 - 2 r H_0) du^2 . \quad (3.10)$$

The Ricci tensor calculated with the metric given by this line-element is given by

$$\hat{R}_{ab} = -P_0^4 |W|^2 k_a k_b , \quad (3.11)$$

on account of (3.6). Thus all of the classical ingredients of high frequency gravity waves described at the end of section 2 are satisfied by these inhomogeneous plane waves for small  $\lambda$ .

## 4 Expansion-free Waves with Cosmological Constant

The classic study of expansion-free gravitational waves with a cosmological constant is the paper by Ozsváth, Robinson and Rózga [20] (for studies of these space-times and their geometric subclasses see [21]–[24]). We will rely heavily on this work in our discussion of high frequency waves of this type. The line-element of interest to us is a generalization of (3.1) given by

$$ds^2 = -2P_\lambda^{-2} \left| d\zeta + \lambda P_\lambda^2 \bar{W}(\bar{\zeta}, u) \sin \frac{u}{\lambda} d\bar{\zeta} \right|^2 + 2 B_\lambda du \left( dr + \frac{1}{2} c_\lambda du \right) , \quad (4.1)$$

with  $P_\lambda, B_\lambda$  real-valued functions of  $\zeta, \bar{\zeta}$  and  $u$ ,  $c_\lambda$  a real-valued function of  $\zeta, \bar{\zeta}$ ,  $r$  and  $u$  and  $W(\zeta, u)$  a complex-valued analytic function of  $\zeta$  and  $u$ . For  $\lambda > 0$  small we require the metric tensor given by this line-element to satisfy approximately Einstein's vacuum field equations with a cosmological constant  $\Lambda$ :

$$R_{ab} = \Lambda g_{ab} + O(\lambda) . \quad (4.2)$$

The detailed results of the calculations of the Ricci tensor here are listed for convenience in appendix B. Together with (4.2) they give us the following:

$$\frac{\partial}{\partial \zeta} \left( P_\lambda^2 \frac{\partial B_\lambda^{1/2}}{\partial \zeta} \right) = 0 , \quad (4.3)$$

and

$$\Delta_\lambda \log P_\lambda - \frac{1}{2} B_\lambda^{-1} \Delta_\lambda B_\lambda + \frac{1}{2} B_\lambda^{-2} P_\lambda^2 \left| \frac{\partial B_\lambda}{\partial \zeta} \right|^2 = \Lambda . \quad (4.4)$$

The function  $c_\lambda$  takes the form

$$c_\lambda = h_\lambda(u) r^2 + f_\lambda(\zeta, \bar{\zeta}, u) r + g_\lambda(\zeta, \bar{\zeta}, u) , \quad (4.5)$$

with

$$h_\lambda(u) = -B_\lambda \Lambda - \frac{1}{2} \Delta_\lambda B_\lambda , \quad (4.6)$$

$$\frac{\partial f_\lambda}{\partial \zeta} = -2 \frac{\partial H_\lambda}{\partial \zeta} + \frac{\partial^2}{\partial u \partial \zeta} (\log B_\lambda) + 2 H_\lambda \frac{\partial}{\partial \zeta} (\log B_\lambda) , \quad (4.7)$$

$$\Delta_\lambda f_\lambda + P_\lambda^2 \left( \frac{\partial}{\partial \bar{\zeta}} (\log B_\lambda) \frac{\partial f_\lambda}{\partial \zeta} + \frac{\partial}{\partial \zeta} (\log B_\lambda) \frac{\partial f_\lambda}{\partial \bar{\zeta}} \right) + 4 B_\lambda^{-1} H_\lambda h_\lambda = 0 , \quad (4.8)$$

and

$$\begin{aligned} -\frac{1}{4} \Delta_\lambda g_\lambda &= -\frac{1}{4} P_\lambda^2 B_\lambda^{-1} \left( \frac{\partial B_\lambda}{\partial \zeta} \frac{\partial g_\lambda}{\partial \zeta} + \frac{\partial B_\lambda}{\partial \zeta} \frac{\partial g_\lambda}{\partial \bar{\zeta}} \right) - \frac{1}{2} B_\lambda^{-1} H_\lambda f_\lambda \\ &= B_\lambda^{-1} \dot{H}_\lambda + B_\lambda^{-1} H_\lambda^2 + B_\lambda^{-2} \dot{B}_\lambda H_\lambda + B_\lambda^{-1} P_\lambda^4 |W|^2 \sin^2 \frac{u}{\lambda} \\ &= 0 . \end{aligned} \quad (4.9)$$

Differentiating (4.4) with respect to  $u$  results in

$$\Delta_\lambda H_\lambda + 2 \Lambda H_\lambda = P_\lambda^2 \frac{\partial}{\partial u} \left( B_\lambda^{-1} \frac{\partial^2 B_\lambda}{\partial \zeta \partial \bar{\zeta}} - \frac{1}{2} B_\lambda^{-2} \frac{\partial B_\lambda}{\partial \zeta} \frac{\partial B_\lambda}{\partial \bar{\zeta}} \right) , \quad (4.10)$$

while the integrability of (4.7) gives us

$$\frac{\partial B_\lambda}{\partial \zeta} \frac{\partial H_\lambda}{\partial \zeta} = \frac{\partial B_\lambda}{\partial \bar{\zeta}} \frac{\partial H_\lambda}{\partial \bar{\zeta}} . \quad (4.11)$$

Now substituting for  $h_\lambda(u)$  from (4.6) and for  $\partial f_\lambda / \partial \zeta$  from (4.7) into (4.8) shows that (4.8) is automatically satisfied on account of (4.10) and (4.11). Before returning to (4.11) we begin by solving (4.4). We are initially guided

by the special case  $\Lambda = 0$  and the expression for  $P_\lambda$  in that case given in the discussion following (3.9), and also by the results in [20]. We take

$$P_\lambda = \left(1 + \frac{\Lambda}{6} |w_\lambda|^2\right) |w'_\lambda|^{-1} , \quad (4.12)$$

where  $w_\lambda(\zeta, u)$  is an analytic function and  $w'_\lambda = \partial w_\lambda / \partial \zeta$ . In addition we shall assume that  $B_\lambda = B_\lambda(w_\lambda, \bar{w}_\lambda, u)$ . Now (4.3) reads

$$\frac{\partial}{\partial w_\lambda} \left( \left(1 + \frac{\Lambda}{6} |w_\lambda|^2\right)^2 \frac{\partial B_\lambda^{1/2}}{\partial w_\lambda} \right) = 0 , \quad (4.13)$$

while (4.4) takes the form

$$\left(1 + \frac{\Lambda}{6} |w_\lambda|^2\right)^2 \frac{\partial^2 B_\lambda^{1/2}}{\partial w_\lambda \partial \bar{w}_\lambda} + \frac{1}{3} \Lambda B_\lambda^{1/2} = 0 . \quad (4.14)$$

These equations are satisfied by the Ozsváth-Robinson-Rózga [20] function

$$B_\lambda^{1/2} = \left(1 + \frac{\Lambda}{6} |w_\lambda|^2\right)^{-1} \left\{ \alpha(u) \left(1 - \frac{\Lambda}{6} |w_\lambda|^2\right) + \beta(u) \bar{w}_\lambda + \bar{\beta}(u) w_\lambda \right\} , \quad (4.15)$$

with  $\alpha$  an arbitrary real-valued function of  $u$  and  $\beta$  an arbitrary complex-valued function of  $u$ . Substitution into (4.6) yields

$$h_\lambda = -\frac{1}{3} \alpha^2 \Lambda - 2 \beta \bar{\beta} = -\kappa(u) \text{ (say)} . \quad (4.16)$$

Referring to the discussion of the function  $P_\lambda(\zeta, \bar{\zeta}, u)$  following (3.9) we can obtain the corresponding waves here with  $\Lambda \neq 0$  by assuming that for small  $\lambda > 0$  we can write

$$w_\lambda(\zeta, u) = \zeta + \lambda^2 g_2 \left( \frac{u}{\lambda} \right) \mathcal{G}_2(\zeta, u) + \dots . \quad (4.17)$$

It follows from this that  $H_\lambda = O(\lambda)$  and  $\dot{H}_\lambda = O(\lambda^0)$ . This circumvents the integrability condition (4.11) because the field equation (4.7) is now replaced by

$$\frac{\partial f_\lambda}{\partial \zeta} = \frac{\partial^2}{\partial \zeta \partial u} (\log B_\lambda) . \quad (4.18)$$

We note that  $O(\lambda)$ -terms are neglected in all field equations as a consequence of the assumption (4.2). We take as solution of (4.18) the function

$$f_\lambda = \frac{\partial}{\partial u} (\log B_\lambda) . \quad (4.19)$$



Now by (4.12) we have

$$P_\lambda = p + O(\lambda^2) , \quad p = 1 + \frac{\Lambda}{6} \zeta \bar{\zeta} , \quad (4.20)$$

and by (4.15),

$$B_\lambda^{1/2} = p^{-1} q + O(\lambda^2) , \quad q = \alpha \left( 1 - \frac{\Lambda}{6} \zeta \bar{\zeta} \right) + \beta \bar{\zeta} + \bar{\beta} \zeta . \quad (4.21)$$

The final field equation (always neglecting  $O(\lambda)$ -terms) is (4.9). Writing  $S_\lambda = B_\lambda^{1/2} g_\lambda = p^{-1} q g_\lambda$  (approximately), this last field equation becomes:

$$-\frac{1}{4} p^{-1} q \left( \Delta S_\lambda + \frac{2}{3} \Lambda S_\lambda \right) - \dot{H}_\lambda + p^4 |W|^2 \sin^2 \frac{u}{\lambda} = 0 , \quad (4.22)$$

with

$$\Delta S_\lambda = 2 p^2 \frac{\partial^2 S_\lambda}{\partial \zeta \partial \bar{\zeta}} . \quad (4.23)$$

Using the argument outlined in Appendix A we find that (4.20) can be made more accurate to read

$$P_\lambda = p + \frac{\lambda^2}{8} p^5 |W|^2 \cos \frac{2u}{\lambda} + O(\lambda^2) , \quad (4.24)$$

and  $S_0 = \lim_{\lambda \rightarrow 0} S_\lambda$  satisfies the differential equation

$$\frac{\partial^2 S_0}{\partial \zeta \partial \bar{\zeta}} + \frac{1}{3} p^{-2} \Lambda S_0 = p^3 q^{-1} |W|^2 . \quad (4.25)$$

The Weyl tensor of the space-time under consideration has Newman-Penrose components  $\Psi_A = O(\lambda^0)$  for  $A = 0, 1, 2, 3$  and

$$\Psi_4 = \lambda^{-1} p^4 q^{-2} W \sin \frac{u}{\lambda} + O(\lambda^0) , \quad (4.26)$$

which is therefore predominantly the radiative type with propagation direction given by the 1-form  $k_a dx^a = du$ .

Now the line-element (4.1) subdivides into a background and an  $O(\lambda)$ -term with the background given by

$$d\hat{s}^2 = -2 p^{-2} |d\zeta|^2 + 2 p^{-2} q^2 du \left\{ dr + \frac{1}{2} \left( p q^{-1} S_0 + 2 q^{-1} \dot{q} r - \kappa(u) r^2 \right) du \right\} , \quad (4.27)$$

and the Ricci tensor of this space-time takes the form

$$\hat{R}_{ab} = \Lambda g_{ab} - p^4 |W|^2 k_a k_b . \quad (4.28)$$

Removing the high-frequency waves ( by putting  $W = 0$ ) results in this line-element coinciding with the Ozsváth–Robinson–Rózga line- element [20]. The function  $S_0$ , in this case satisfying (4.25) with  $W = 0$ , is required for the description of waves which include generalized Kundt waves. If  $S_0 = 0$  in (4.27) then this is the line-element of the de Sitter universe in a coordinate system based on a family of null hypersurfaces  $u = \text{constant}$  whose generators have vanishing expansion and shear (see Appendix C).

## 5 Comparison with ‘Spherical’ Waves

The gravitational waves emitted from isolated gravitating systems are asymptotically almost spherical. The wave fronts have histories in space-time which are expanding, shear-free null hypersurfaces. The space-time models of the vacuum gravitational fields of such waves in the high frequency approximation, including a cosmological constant  $\Lambda$ , have line-elements which are roughly similar to (3.1). Specifically they are given by [25]

$$ds^2 = -2r^2 p_\lambda^{-2} \left| d\zeta + \frac{\lambda p_\lambda^2}{r} \bar{W}(\bar{\zeta}, u) \sin \frac{u}{\lambda} d\bar{\zeta} \right|^2 + 2 du dr + c_\lambda du^2 . \quad (5.1)$$

Here  $p_\lambda(\zeta, \bar{\zeta}, u)$  is a real-valued function,  $W$  is an arbitrary analytic function and the real-valued function  $c_\lambda(\zeta, \bar{\zeta}, u, r)$  is given by

$$c_\lambda = K_\lambda - 2r H_\lambda - \frac{1}{3} \Lambda r^2 - \frac{2m_\lambda}{r} , \quad (5.2)$$

where

$$K_\lambda = \Delta_\lambda \log P_\lambda , \quad H_\lambda = P_\lambda^{-1} \dot{P}_\lambda , \quad (5.3)$$

with

$$\Delta_\lambda = 2p_\lambda^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} , \quad m_\lambda = m_\lambda(u) . \quad (5.4)$$

Once again the Ricci tensor calculated with the metric given by (5.1) has the form (4.2) for small  $\lambda$  provided the following field equation is satisfied:

$$\dot{m}_\lambda - 3m_\lambda H_\lambda - \frac{1}{4} \Delta_\lambda K_\lambda + p_\lambda^4 |W|^2 \sin^2 \frac{u}{\lambda} = 0 . \quad (5.5)$$

The propagation direction (in the space-time with line-element (5.1)) of the histories of the wave fronts is  $k^a \partial / \partial x^a = \partial / \partial r$  and the integral curves of this vector field have real expansion  $r^{-1}$  and complex shear given by

$$\sigma = \frac{\lambda p_\lambda^2}{r^2} W \sin \frac{u}{\lambda} + O(\lambda^2) = O(\lambda) . \quad (5.6)$$

Thus in the limit  $\lambda \rightarrow 0$  these are expanding, shear-free null geodesics.

The background (corresponding to  $\lambda = 0$ ) Ricci tensor is now given by

$$\hat{R}_{ab} = -r^{-2} p_0^4 |W|^2 k_a k_b + \Lambda \hat{g}_{ab} , \quad (5.7)$$

with  $p_0$  (and  $m_0$ ) satisfying the equation (see Appendix A)

$$\dot{m}_0 - 3 m_0 H_0 - \frac{1}{4} \Delta_0 K_0 + \frac{1}{2} p_0^4 |W|^2 = 0 , \quad (5.8)$$

and the background line-element  $d\hat{s}^2 = \hat{g}_{ab} dx^a dx^b$  has Robinson–Trautman [26] form. For small  $\lambda > 0$  the Riemann curvature tensor calculated with the metric given by the line-element (5.1) is now  $\Psi_A = O(\lambda^0)$  for  $A = 0, 1, 2, 3$  and

$$\Psi_4 = \frac{1}{r} \lambda^{-1} p_0^2 W \sin \frac{u}{\lambda} + O(\lambda^0) . \quad (5.9)$$

To obtain an equation for  $p_\lambda$  which corresponds to (2.6) and (3.5) we first note that the form of the line-element (5.1) is preserved with an  $O(\lambda^2)$ -error (sufficient for our purposes) by a coordinate transformation of the form (a special case of a Robinson–Trautman [26] transformation)

$$u = u' + \lambda^2 w_2 \left( \frac{u'}{\lambda} \right) \gamma(u') + O(\lambda^3) , \quad r = \frac{du}{du'} r' , \quad (5.10)$$

for some real-valued functions  $w_2$  and  $\gamma$ . Under such a transformation the function  $m_\lambda$  is transformed to

$$\hat{m}_\lambda = (1 + 3 \lambda w_2' \gamma + O(\lambda^2)) m_\lambda , \quad (5.11)$$

with the prime on  $w_2$  denoting differentiation with respect to its argument. For small  $\lambda > 0$  we can write (see remark in Appendix A and (A.2))

$$\begin{aligned} m_\lambda(u) &= m_0(u) + \lambda q_1 \left( \frac{u}{\lambda} \right) m_1(u) + O(\lambda^2) , \\ &= m_0(u') + \lambda q_1 \left( \frac{u'}{\lambda} \right) m_1(u') + O(\lambda^2) . \end{aligned} \quad (5.12)$$

Substituting (5.12) into (5.11) we see that we can choose the functions  $w_2$ ,  $\gamma$  in the coordinate transformation (5.10) so as to eliminate the  $O(\lambda)$ -term in  $\hat{m}_\lambda$ . Thus without loss of generality we may take

$$m_\lambda(u) = m_0(u) + O(\lambda^2) , \quad (5.13)$$

in (5.2), (5.4) and (5.5). Now solving (5.5) using (5.8) yields

$$p_\lambda = p_0 - \frac{1}{12} \lambda m_0^{-1} p_0^5 |W|^2 \sin \frac{2u}{\lambda} + O(\lambda^2) . \quad (5.14)$$

## 6 Conclusion

The pattern of the line-elements (2.1), (3.1), (4.1) and (5.1) is now clear and they could reasonably claim to be the basic non-expanding and expanding high frequency gravitational wave solutions of Einstein's field equations. They are useful building blocks for the study of the interaction of such waves in General Relativity. A start in this direction has already been made with the description of the head-on collision of the homogeneous plane waves of section 2 in [27].

The introduction of a cosmological constant is far more complicated in the 'plane wave' case than in the 'spherical wave' case. We have described this for the special case of adding a cosmological constant to high frequency  $pp$ -waves. The introduction of a cosmological constant for the general high frequency waves discussed in section 3 is a topic for further study.

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## A Taking the limit $\lambda \rightarrow 0$

The passage from eq.(2.3) to eq.(2.7) is brought about quite simply as follows: for definiteness let us suppose that the coordinate  $u$  has a finite range  $u_1 \leq u \leq u_2$  during which the radiation is detected. Let  $u'$  be any value of  $u$  within this range and  $\vartheta(u - u')$  be the Heaviside step function which is equal to unity for  $u - u' > 0$  and which vanishes for  $u - u' < 0$ . This function acts as a convenient Greens' function for the equation (2.3). If we multiply (2.3) by  $\vartheta(u - u')$  and integrate the equation with respect to  $u$  in the range  $u_1 \leq u \leq u_2$  then it is easy to see that the resulting integral equation is

$$H_\lambda(u') = H_\lambda(u_2) + \int_{u_1}^{u_2} \left\{ H_\lambda^2(u) + |W(u)|^2 \sin^2 \frac{u}{\lambda} \right\} \vartheta(u - u') du . \quad (\text{A.1})$$

We assume that all functions having a subscript  $\lambda$  have uniform  $\lambda = 0$  limits. If  $\mathcal{F}_\lambda(\zeta, \bar{\zeta}, u)$  is any such real-valued function we assume that for small  $\lambda > 0$  it has an expansion of the Isaacson [12] form:

$$\mathcal{F}_\lambda = \mathcal{F}_0(\zeta, \bar{\zeta}, u) + \lambda f_1\left(\frac{u}{\lambda}\right) \mathcal{F}_1(\zeta, \bar{\zeta}, u) + \lambda^2 f_2\left(\frac{u}{\lambda}\right) \mathcal{F}_2(\zeta, \bar{\zeta}, u) + \dots , \quad (\text{A.2})$$

where  $f_1, f_2$  etc. are real-valued functions. In order to take the limit  $\lambda \rightarrow 0$  of eq.(A.1) we make use of the Riemann–Lebesgue theorem [28] which states that if a real-valued function  $A(u)$  (it could depend on additional variables such as  $\zeta, \bar{\zeta}$ , but its dependence on  $u$  is the critical aspect for the theorem) is integrable (and therefore could, for example, be a step function) on the interval  $u_1 \leq u \leq u_2$  then

$$\lim_{\lambda \rightarrow 0} \int_{u_1}^{u_2} A(u) \sin \frac{u}{\lambda} du = 0 . \quad (\text{A.3})$$

It therefore follows that

$$\lim_{\lambda \rightarrow 0} \int_{u_1}^{u_2} A(u) \sin^2 \frac{u}{\lambda} du = \frac{1}{2} \int_{u_1}^{u_2} A(u) du . \quad (\text{A.4})$$

Thus taking the limit  $\lambda \rightarrow 0$  of (A.1) results in the integral equation

$$H_0(u') = H_0(u_2) + \int_{u_1}^{u_2} \left\{ H_0^2(u) + \frac{1}{2} |W(u)|^2 \right\} \vartheta(u - u') du . \quad (\text{A.5})$$

Differentiating this equation with respect to  $u'$  (and using  $d(\vartheta(u - u'))/du' = \delta(u - u')$ , the Dirac delta function) shows that  $H_0$  satisfies (2.7)

The argument given here, when applied to the dependence of the functions on the coordinate  $u$ , results in the passage from (3.4) to (3.6). In addition writing (5.5) in the form

$$\frac{\partial}{\partial u} (p_\lambda^{-3} m_\lambda) - \frac{1}{4} p_\lambda^{-3} \Delta_\lambda K_\lambda + p_\lambda |W|^2 \sin^2 \frac{u}{\lambda} = 0 , \quad (\text{A.6})$$

and applying the same argument yields (5.8).

## B The Ricci Tensor Required for Section 4

A convenient basis of 1-form fields for the space-time with line-element (4.1) is defined by:

$$\vartheta^1 = P_\lambda^{-1} (d\zeta + \lambda P_\lambda^2 \bar{W} \sin \frac{u}{\lambda} d\bar{\zeta}) = -\vartheta_2 , \quad (\text{B.1})$$

$$\vartheta^2 = \bar{\vartheta}^1 = -\vartheta_1 , \quad (\text{B.2})$$

$$\vartheta^3 = B_\lambda du = \vartheta_4 , \quad (\text{B.3})$$

$$\vartheta^4 = dr + \frac{1}{2} c_\lambda du = \vartheta_3 . \quad (\text{B.4})$$

The components of the Ricci tensor calculated on the half-null tetrad given via these 1-forms are:

$$R_{11} = \bar{R}_{22} = B_\lambda^{-1} \frac{\partial}{\partial \zeta} \left( P_\lambda^2 B_\lambda^{-1/2} \frac{\partial B_\lambda}{\partial \zeta} \right) + O(\lambda) , \quad (\text{B.5})$$

$$R_{12} = -\Delta_\lambda \log P_\lambda + \frac{1}{2} B_\lambda^{-1} \left( \Delta_\lambda B_\lambda - B_\lambda^{-1} P_\lambda^2 \left| \frac{\partial B_\lambda}{\partial \zeta} \right|^2 \right) + O(\lambda) , \quad (\text{B.6})$$

$$\begin{aligned} R_{13} = \bar{R}_{23} = & -P_\lambda \frac{\partial}{\partial \zeta} (B_\lambda^{-1} H_\lambda) + \frac{1}{2} B_\lambda^{-1} P_\lambda \frac{\partial^2}{\partial u \partial \zeta} (\log B_\lambda) \\ & - \frac{1}{2} B_\lambda^{-1} P_\lambda \frac{\partial^2 c_\lambda}{\partial r \partial \zeta} + O(\lambda) , \end{aligned} \quad (\text{B.7})$$

$$R_{34} = -\frac{1}{2} B_\lambda^{-1} \Delta_\lambda B_\lambda - \frac{1}{2} B_\lambda^{-1} \frac{\partial^2 c_\lambda}{\partial r^2} + O(\lambda) , \quad (\text{B.8})$$

$$\begin{aligned} R_{33} = & -\frac{1}{2} B_\lambda^{-1} \Delta_\lambda c_\lambda - \frac{1}{2} P_\lambda^2 B_\lambda^{-2} \left\{ \frac{\partial B_\lambda}{\partial \bar{\zeta}} \frac{\partial c_\lambda}{\partial \zeta} + \frac{\partial B_\lambda}{\partial \zeta} \frac{\partial c_\lambda}{\partial \bar{\zeta}} \right\} \\ & - B_\lambda^{-2} H_\lambda \frac{\partial c_\lambda}{\partial r} - 2 B_\lambda^{-2} \dot{H}_\lambda + 2 B_\lambda^{-2} H_\lambda^2 \\ & + 2 B_\lambda^{-3} \dot{B}_\lambda H_\lambda + 2 B_\lambda^{-2} P_\lambda^4 |W|^2 \sin^2 \frac{u}{\lambda} + O(\lambda) , \end{aligned} \quad (\text{B.9})$$

$$R_{14} = \bar{R}_{24} = O(\lambda) , \quad (\text{B.10})$$

$$R_{44} = O(\lambda) . \quad (\text{B.11})$$

Here as before

$$H_\lambda = P_\lambda^{-1} \dot{P}_\lambda , \quad \Delta_\lambda = 2 P_\lambda^2 \frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} , \quad (\text{B.12})$$

and the dot indicates partial differentiation with respect to  $u$ .

## C The de Sitter Line-Element

The form of the line-element of de Sitter space-time given in [20] is of paramount importance in the context of gravitational wave theory since it is based on a family of null hyperplanes (generated by null geodesics with zero expansion and shear) in the space-time. Since it is well known (see for example [29]) that the de Sitter universe can be viewed as a quadric  $V_4$  in 5-dimensional Minkowskian space-time  $V_5$  we will demonstrate how the Ozsváth–Robinson–Rózga form of the de Sitter line-element emerges from this point of view (this is discussed in a slightly different and separate form in [30]). The line-element of  $V_5$  is taken to be

$$-ds^2 = \frac{3}{\Lambda} (dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 - (dX^4)^2 , \quad (\text{C.1})$$

and we assume that  $\Lambda \neq 0$  ( $\Lambda$  is the cosmological constant;  $V_4$  is a space-time of constant curvature  $\Lambda/3$ ). The equation of the quadric  $V_4$  is given by

$$\frac{3}{\Lambda} (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = \frac{3}{\Lambda} . \quad (\text{C.2})$$

The induced line-element on  $V_4$  is got by restricting (C.1) to  $V_4$ .

A useful shorthand for (C.1) and (C.2) is to write them in an obvious vector notation [29] as

$$-ds^2 = d\mathbf{X} \cdot d\mathbf{X} , \quad \mathbf{X} \cdot \mathbf{X} = \frac{3}{\Lambda} . \quad (\text{C.3})$$

Tran [31] has shown that the intersection of the quadric  $V_4$  with the null hyperplane passing through the origin of  $V_5$ ,

$$\mathbf{b} \cdot \mathbf{X} = 0 , \quad \mathbf{b} \cdot \mathbf{b} = 0 , \quad (\text{C.4})$$

is a null hyperplane (whose geodesic generators have vanishing expansion and shear) in  $V_4$ .

We can parametrise the points on  $V_4 \subset V_5$  using  $(\zeta, \bar{\zeta}, r, u)$  by writing the position vector of a point on  $V_5$  in the form

$$\mathbf{X} = \mathbf{Y}(\zeta, \bar{\zeta}, u) + p^{-1} q r \mathbf{a}(u) , \quad (\text{C.5})$$



with

$$Y^0 = p^{-1} \left( \frac{\Lambda}{6} \zeta \bar{\zeta} - 1 \right) , \quad (C.6)$$

$$Y^1 + iY^2 = p^{-1} \sqrt{2} \zeta , \quad (C.7)$$

$$Y^3 = Y^4 = p^{-1} \left\{ \bar{l}(u) \zeta + l(u) \bar{\zeta} + m(u) \left( 1 - \frac{\Lambda}{6} \zeta \bar{\zeta} \right) \right\} , \quad (C.8)$$

with  $p = 1 + \frac{\Lambda}{6} \zeta \bar{\zeta}$ ,

$$a^0(u) = -\frac{\Lambda}{3} m(u) , \quad (C.9)$$

$$a^1(u) + i a^2(u) = \sqrt{2} l(u) , \quad (C.10)$$

$$a^3(u) - a^4(u) = -1 , \quad (C.11)$$

$$a^3(u) + a^4(u) = \frac{\Lambda}{3} m^2 + 2 l \bar{l} , \quad (C.12)$$

and

$$q = \bar{\beta}(u) \zeta + \beta(u) \bar{\zeta} + \alpha(u) \left( 1 - \frac{\Lambda}{6} \zeta \bar{\zeta} \right) , \quad (C.13)$$

with  $\beta(u) = \dot{l}(u)$ ,  $\alpha(u) = \dot{m}(u)$ . It follows from these equations that

$$\mathbf{Y} \cdot \mathbf{Y} = \frac{3}{\Lambda} , \quad \mathbf{a} \cdot \mathbf{Y} = 0 , \quad \mathbf{a} \cdot \mathbf{a} = 0 . \quad (C.14)$$

Thus  $\mathbf{Y}$  is a point on  $V_4$  corresponding to  $r = 0$  and, more importantly,  $\mathbf{X}$  in (C.5) is a point on  $V_4$ . Substituting (C.5) into (C.3) gives the induced line-element on de Sitter space due to its embedding in  $V_5$ . This calculation is aided with the following scalar products:

$$\mathbf{a} \cdot \frac{\partial \mathbf{Y}}{\partial u} = -p^{-1} q , \quad \frac{\partial \mathbf{Y}}{\partial u} \cdot \frac{\partial \mathbf{Y}}{\partial u} = 0 , \quad \dot{\mathbf{a}} \cdot \frac{\partial \mathbf{Y}}{\partial u} = 0 . \quad (C.15)$$

We also have  $\dot{\mathbf{a}} \cdot \dot{\mathbf{a}} = \frac{\Lambda}{3} \alpha^2 + 2 \beta \bar{\beta}$  and the further scalar products:

$$\frac{\partial \mathbf{Y}}{\partial \zeta} \cdot \frac{\partial \mathbf{Y}}{\partial \zeta} = p^{-2} , \quad \frac{\partial \mathbf{Y}}{\partial \zeta} \cdot \frac{\partial \mathbf{Y}}{\partial \bar{\zeta}} = 0 = \frac{\partial \mathbf{Y}}{\partial \bar{\zeta}} \cdot \frac{\partial \mathbf{Y}}{\partial \zeta} , \quad \dot{\mathbf{a}} \cdot \frac{\partial \mathbf{Y}}{\partial \zeta} = \frac{\partial}{\partial \zeta} (p^{-1} q) . \quad (C.16)$$

This calculation results in the line-element

$$ds^2 = -2 p^{-2} d\zeta d\bar{\zeta} + 2 p^{-2} q^2 du dr + p^{-2} q^2 (2 q^{-1} \dot{q} r - \kappa(u) r^2) du^2 , \quad (C.17)$$

with  $\kappa(u) = 2 \beta \bar{\beta} + \Lambda \alpha^2 / 3$ , which is the line-element of de Sitter space in the interesting form given by Ozsváth, Robinson and Rózga [20]. Using the

construction of de Sitter space as the quadric (C.2), Tran [31] has provided beautiful proofs that properties of the functions  $\alpha(u), \beta(u)$  and  $\bar{\beta}(u)$ , along with the sign of the cosmological constant  $\Lambda$ , determine whether or not the null hypersurfaces  $u = \text{constant}$  intersect [Tran's results [31] are summarized as follows: if  $\Lambda > 0$  then  $\kappa > 0$  implies intersections while if  $\Lambda < 0$  then (a)  $\kappa < 0$  implies intersections, (b)  $\kappa = 0$  implies intersections except if  $\text{Im}\beta = 0$  or  $\text{Re}\beta = 0$  or  $\text{Re}\beta = C \text{Im}\beta$  for some constant  $C$  and (c)  $\kappa > 0$  implies intersections].